

# Self-propulsion velocity of $N$ -sphere micro-robot

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The aim of this paper is to derive an analytical expression for the self-propulsion velocity of a micro-swimmer that consists of  $N$  spheres moving along a fixed line. The spheres are linked to each other by the rods of the prescribed lengths changing periodically. For the derivation we use the asymptotic procedure containing the two-timing method and a distinguished limit. Our final formula shows that in the main approximation the self-propulsion velocity is determined by the interactions between all possible triplets of spheres.

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## 1. Introduction and formulation of problem

### 1.1. *Introduction*

The studies of simple micro-swimmers (or micro-robots) represent a flourishing modern research topic (see Purcell (1977), Alexander, Pooley and Yeomans (2009), Earl, Pooley, Ryder, Bredberg and Yeomans (2010), Alouges, DeSimone and Lefebvre (2008), Becker, Koelher, and Stone (2003), Gilbert, Ogrin, Petrov, and Winlove (2010), Golestanian & Ajdari (2008), Golestanian & Ajdari (2009)), which creates the fundamental base for modern applications in medicine and other areas. In this paper we generalize the three-sphere micro-swimmer by Golestanian & Ajdari (2008) to the general

$N$ -sphere micro-swimmer. It is possible since the employed two-timing method and a distinguished limit significantly simplifies the analytical procedure.

### 1.2. Formulation of problem

We consider a micro-swimmer consisting of  $N$  rigid spheres of radii  $R_i^*$ ,  $i = 1, 2, \dots, N$  with their centers at the points  $x_i^*(t^*)$  of the  $x^*$ -axis ( $x_{i+1}^* > x_i^*$ );  $t^*$  is time. The spheres are connected by  $N - 1$  rods of lengths  $l_i^{k*} = x_k^* - x_i^*$  where our choice is always  $k > i$ . The masses of the spheres and the rods (in the Stokes approximation) are zero. The rods are so thin that their interaction with a fluid is negligible. The lengths of the rods are prescribed as

$$l_i^{k*} = L_i^{k*} + \widetilde{\lambda}_i^{k*} \quad (1.1)$$

where  $L_i^{k*}$  are the averaged values and  $\widetilde{\lambda}_i^{k*}(\tau)$  are the oscillations, which are prescribed as  $2\pi$ -periodic functions of  $\tau \equiv \omega t^*$  with a constant frequency  $\omega$ . Asterisks mark dimensional variables and parameters.

In the Stokes approximation the total force acting on each sphere is zero (their masses are zero), hence the equation of motion for the  $i$ -th sphere can be written as

$$\kappa_i^* \dot{x}_i^* - \sum_{k \neq i} 3\kappa_i^* R_k^* \dot{x}_k^* / (2l_i^{k*}) = -f_i^* \quad (1.2)$$

where  $\kappa_i^* \equiv 6\pi\eta R_i^*$ ,  $\eta$  is viscosity, dots above the functions stands for  $d/dt^*$ . The l.h.s. of (1.2) represents a viscous friction, while  $f_i^*$  is the force exerted by the rods to the  $i$ -th sphere. In order to derive (1.2) we have used the fact (see Lamb (1932), Landau & Lifshitz (1959), Moffatt (1996)) that a sphere of radius  $R_k^*$  and position  $x_k^*$  moving along the  $x^*$ -axis with velocity  $\dot{x}_k^*$  creates at the center of  $i$ -th sphere the  $x^*$ -component of fluid velocity equal to  $-3R_k^* \dot{x}_k^* / (2l_i^{k*})$ , where the minus sign corresponds to  $x_k > x_i$ . The considered mechanical system is a closed one, hence the total force exerted

by the constraints is zero:

$$\sum_{i=1}^N f_i^* = 0 \quad (1.3)$$

Eqns. (1.2),(1.3) represent the system of ODEs to be solved in this paper. Notice that we do not use the summation convention.

The equation (1.2) and its solution  $\mathbf{x}^*(t^*) = (x_1^*, x_2^*, \dots, x_N^*)$ , contain three characteristic lengths: the radius  $R$  of spheres, the distance  $L$  between the neighbouring spheres, and the amplitude  $\lambda$  of oscillations of rod lengths; at the same time the only explicit characteristic time-scale  $T$  corresponds to the frequency  $\omega$ :

$$R, \quad L, \quad \lambda, \quad T \equiv 1/\omega \quad (1.4)$$

The dimensionless variables and small parameters are

$$\mathbf{x}^* = L\mathbf{x}, \quad L_{ik}^* = LL_i^k, \quad R_i^* = RR_i, \quad \tilde{\lambda}_i^{k*} = \lambda\tilde{\lambda}_i^k, \quad t^* = Tt \quad (1.5)$$

$$f_i^* = -6\pi\eta RLf_i/T, \quad \varepsilon \equiv \lambda/L \ll 1, \quad \delta \equiv 3R/(2L) \ll 1$$

Then the dimensionless eqns.(1.1)-(1.3) take the form

$$R_i \dot{x}_i - \delta \sum_{k \neq i} R_{ik} \dot{x}_k / l_{ik} = f_i, \quad R_{ik} \equiv R_i R_k \quad (1.6)$$

$$l_{ik} = L_{ik} + \varepsilon \tilde{\lambda}_{ik} \quad (1.7)$$

$$\sum_i f_i = \mathbf{f} \cdot \mathbf{I} = 0, \quad \mathbf{I} \equiv (1, 1, \dots, 1) \quad (1.8)$$

One should note that ‘dots’ above function in (1.2) and (1.6) correspond to the dimensional and dimensionless time derivatives correspondingly. The first equation (1.6) can be rewritten in the matrix form

$$\mathbb{A} \dot{\mathbf{x}} = \mathbf{f} \quad \text{or} \quad \sum_{k=1}^N A_{ik} \dot{x}_k = f_i \quad (1.9)$$

$$\mathbb{A} = A_{ik} = \begin{cases} R_i & \text{for } i = k, \\ -\delta R_{ik}/l_{ik} & \text{for } i \neq k \end{cases} \quad (1.10)$$

## 1.3. Notations

The variables  $\mathbf{x} = (x_1, x_2, \dots, x_N)$ ,  $t$ ,  $s$ , and  $\tau$  serve as dimensionless coordinates of spheres, physical time, slow time, and fast time. We use the following definitions and notations:

(i) A dimensionless function  $f = f(s, \tau)$  belongs to the class  $\mathfrak{D}(1)$  if  $f = O(1)$  and all partial  $s$ -, and  $\tau$ -derivatives of  $f$  (required for our consideration) are also  $O(1)$ . In this paper all small parameters appear as explicit multipliers, while all functions always belong to  $\mathfrak{D}(1)$ -class.

(ii) We consider only *periodically oscillating in  $\tau$  functions*

$$f \in \mathfrak{H} : \quad f(s, \tau) = f(s, \tau + 2\pi) \quad (1.11)$$

where the  $s$ -dependence is not specified. Hence  $f \in \mathfrak{H} \cap \mathfrak{D}(1)$ .

(iii) The subscripts  $\tau$  and  $s$  denote the related partial derivatives.

(iv) For an arbitrary  $f \in \mathfrak{H}$  the *averaging operation* is

$$\langle f \rangle \equiv \frac{1}{2\pi} \int_{\tau_0}^{\tau_0 + 2\pi} f(s, \tau) d\tau, \quad \forall \tau_0 \quad (1.12)$$

where during the integration we keep  $s = \text{const}$  and  $\langle f \rangle$  does not depend on  $\tau_0$ .

(v) The class of *tilde-functions* (or purely oscillating functions) is such that

$$\tilde{f} : \quad \tilde{f}(s, \tau) = \tilde{f}(s, \tau + 2\pi), \quad \text{with} \quad \langle \tilde{f} \rangle = 0. \quad (1.13)$$

Tilde-functions represent a special case of  $\mathfrak{H}$ -functions with zero average.

(vi) The class of *bar-functions* (or mean-functions) is defined as

$$\overline{f} : \quad \overline{f}_\tau \equiv 0, \quad \overline{f}(s) = \langle \overline{f}(s) \rangle \quad (1.14)$$

## 2. Asymptotic procedure

The use of  $\varepsilon$ -dependence of  $l_{ik}$  (1.1) leads to the presentation of matrix  $\mathbb{A}$  (1.10) as a series for  $\varepsilon \rightarrow 0$  (we consider  $\delta$  as a fixed parameter)

$$\begin{aligned} \mathbb{A} &= \overline{\mathbb{C}} + \varepsilon \delta \tilde{\mathbb{A}}'_0 + \dots, \quad \overline{\mathbb{C}}_0 \equiv \overline{\mathbb{A}}_0 + \delta \overline{\mathbb{B}}_0 \\ \overline{\mathbb{A}}_0 &\equiv \text{diag}\{R_1, R_2, \dots, R_N\}, \quad \tilde{\mathbb{A}}'_0 \equiv \begin{cases} 0 & \text{for } i = k, \\ R_{ik} \tilde{\lambda}_{ik} / L_{ik}^2 & \text{for } i \neq k \end{cases} \end{aligned} \quad (2.1)$$

where we do not present the expression for  $\overline{\mathbb{B}}_0$  since it will not affect the answer.

The crucial step of our procedure is the introduction of a fast time variable  $\tau$  and a slow time variable  $s$ . We take  $\tau = t$  (which corresponds to the prescribed oscillations of the rods) and  $s = \varepsilon^2 t$ . This choice of  $s$  can be justified by the same distinguished limit arguments as in Vladimirov (2012); here we present this fact without proof, referring only to the most important fact that it leads to a valid asymptotic procedure. Therefore we use the chain rule  $d/dt = \partial/\partial\tau + \varepsilon^2 \partial/\partial s$  and then we accept (temporarily) that  $\tau$  and  $s$  represent two independent variables. The two-timing form of eqn. (1.9) is

$$(\overline{\mathbb{C}}_0 + \varepsilon \delta \tilde{\mathbb{A}}'_0 + \dots)(\mathbf{x}_\tau + \varepsilon^2 \mathbf{x}_s) = \mathbf{f} \quad (2.2)$$

where unknown functions are taken as the series

$$\mathbf{x}(\tau, s) = \overline{\mathbf{x}}_0(s) + \varepsilon \mathbf{x}_1(\tau, s) + \dots, \quad \mathbf{f}(\tau, s) = \mathbf{f}_0(\tau, s) + \varepsilon \mathbf{f}_1(\tau, s) + \dots \quad (2.3)$$

The accepted condition  $\tilde{\mathbf{x}}_0 \equiv 0$  reflects the fact that the large distances of self-swimming are driven by small self-oscillations. Now we consider the successive approximations of (2.2), (2.3) in  $\varepsilon$ :

- (i) *Terms  $O(\varepsilon^0)$*  give  $\mathbf{f}_0 \equiv 0$ .
- (ii) *Terms  $O(\varepsilon^1)$*  give  $\mathbb{C}_0 \mathbf{x}_{0\tau} = \mathbf{f}_1$ ; the averaged part of this equation gives  $\mathbf{f}_1 \equiv 0$ , while

the oscillating part yields

$$\mathbb{C}_0 \tilde{\mathbf{x}}_{1\tau} = \tilde{\mathbf{f}}_1 \quad (2.4)$$

(iii) *Terms*  $O(\varepsilon^2)$  give the equation  $\overline{\mathbb{C}}_0 \tilde{\mathbf{x}}_{2\tau} + \delta \tilde{\mathbb{A}}'_0 \tilde{\mathbf{x}}_{1\tau} + \overline{\mathbb{C}}_0 \overline{\mathbf{x}}_{0s} = \mathbf{f}_2$ ; its averaged part is

$$\overline{\mathbb{C}}_0 \overline{\mathbf{x}}_{0s} + \delta \langle \tilde{\mathbb{A}}'_0 \tilde{\mathbf{x}}_{1\tau} \rangle = \overline{\mathbf{f}}_2 \quad (2.5)$$

The force  $\overline{\mathbf{f}}_2$  can be excluded from (2.5) by (1.8):

$$\mathbf{I} \cdot \overline{\mathbb{C}}_0 \overline{\mathbf{x}}_{0s} + \delta \mathbf{I} \cdot \langle \tilde{\mathbb{A}}'_0 \tilde{\mathbf{x}}_{1\tau} \rangle = 0 \quad (2.6)$$

The self-propulsion with averaged velocity  $\overline{V}_0$  means that  $\mathbf{x}_{0s} = \overline{V}_0 \mathbf{I}$ , hence

$$\overline{V}_0 = -\delta \frac{\mathbf{I} \cdot \langle \tilde{\mathbb{A}}'_0 \tilde{\mathbf{x}}_{1\tau} \rangle}{\mathbf{I} \cdot \overline{\mathbb{A}}_0 \mathbf{I}} \quad (2.7)$$

where the matrix  $\overline{\mathbb{C}}_0$  is replaced with  $\overline{\mathbb{A}}_0$  since we consider only the main (linear) term in  $\delta$ . Expression (2.7) still contains unknown functions  $\tilde{\mathbf{x}}_{1\tau}$  which can be determined from (2.4) with the use of constrains (1.7),(1.8). Indeed, the equation (2.4) (with linear in  $\delta$  precision) gives  $\tilde{\mathbf{x}}_{1\tau} = \mathbb{A}_0^{-1} \tilde{\mathbf{f}}_1$  with  $\mathbb{A}_0^{-1} = \text{diag}\{1/R_1, 1/R_2, \dots, 1/R_N\}$ ; it means that  $\tilde{\mathbf{x}}_{1\tau} = \tilde{\mathbf{g}}$  with the components  $\tilde{g}_i \equiv \tilde{f}_{1i}/R_i$ . One can see that (1.7) yields  $\tilde{g}_k - \tilde{g}_i = \tilde{\lambda}_{i\tau}^k$ , while (1.8) leads to  $\sum_i R_i \tilde{g}_i = 0$ . Both these restrictions can be written as one  $N \times N$  matrix equation

$$\mathbb{M} \tilde{\mathbf{g}} = \tilde{\mathbf{l}}_\tau, \quad \mathbb{M} \equiv \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & -0 & \dots & -1 & 1 \\ R_1 & R_2 & R_3 & \dots & R_{N-1} & R_N \end{pmatrix}, \quad \tilde{\mathbf{l}} \equiv \begin{pmatrix} \tilde{\lambda}_1^2 \\ \tilde{\lambda}_2^3 \\ \dots \\ \tilde{\lambda}_{N-1}^N \\ 0 \end{pmatrix} \quad (2.8)$$

The substitution of  $\tilde{\mathbf{x}}_{1\tau} = \mathbb{M}^{-1} \tilde{\mathbf{l}}_\tau$  into (2.7) gives us self-propulsion velocity in the matrix form

$$\overline{V}_0 = -\delta \frac{\mathbf{I} \cdot \langle \tilde{\mathbb{A}}'_0 \mathbb{M}^{-1} \tilde{\mathbf{l}}_\tau \rangle}{\mathbf{I} \cdot \overline{\mathbb{A}}_0 \mathbf{I}} \quad (2.9)$$

where the matrix  $\mathbb{M}^{-1}$  is

$$(-1)^{N+1} \Delta \mathbb{M}^{-1} \equiv \begin{pmatrix} \Delta_1 - \Delta & \Delta_2 - \Delta & \Delta_3 - \Delta & \dots & \Delta_{N-1} - \Delta & 1 \\ \Delta_1 & \Delta_2 - \Delta & \Delta_3 - \Delta & \dots & \Delta_{N-1} - \Delta & 1 \\ \Delta_1 & \Delta_2 & \Delta_3 - \Delta & \dots & \Delta_{N-1} - \Delta & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \Delta_1 & \Delta_2 & \Delta_3 & \dots & \Delta_{N-1} - \Delta & 1 \\ \Delta_1 & \Delta_2 & \Delta_3 & \dots & \Delta_{N-1} & 1 \end{pmatrix} \quad (2.10)$$

$$\Delta_k \equiv \sum_{\alpha=1}^k R_\alpha, \quad k \geq 1; \quad \Delta \equiv \Delta_N.$$

Further calculations show that (2.9) can be presented as

$$\overline{V}_0 = \frac{\delta}{\Delta^2} \sum_{i < k < l} \overline{G}_{ikl} \quad (2.11)$$

$$\overline{G}_{ikl} \equiv R_i R_k R_l \left( \frac{1}{L_{ik}^2} + \frac{1}{L_{kl}^2} - \frac{1}{L_{il}^2} \right) \langle \tilde{\lambda}_{ik} \tilde{\lambda}_{kl\tau} - \tilde{\lambda}_{ik\tau} \tilde{\lambda}_{kl} \rangle$$

where the sum is taken over all possible triplets  $(i, k, l) : 1 \leq i < k < l \leq N$ .

For the three-swimmer this sum contains the only term, which coincides with one by Golestanian & Ajdari (2008). In general it contains  $N!/[(N-3)!3!]$  terms: for the four-swimmer we already have four triplets  $(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)$ , for the five-swimmer – 10 terms, while for the ten-swimmer the number of triplets grows up to 120.

The expressions for  $\mathbb{M}^{-1}$  (2.10) and  $\overline{V}_0$  (2.11) have been obtained by the explicit calculations for  $N = 3, 4, 5$  and by the mathematical induction for any  $N$ .

Formula (2.11) represents the main result of this paper. According to (2.11)  $\mathbf{I} \cdot \overline{\mathbf{x}}_s = \overline{V}_0 = O(\delta)$ ; however physical velocity is  $\mathbf{I} \cdot \overline{\mathbf{x}}_t = \varepsilon^2 \overline{V}_0$ . Hence the order of magnitude of the dimensionless physical velocity is  $O(\varepsilon^2 \delta)$ .

### 3. Discussion

1. The explicit formula (2.11) allows one to find the optimal strokes, to calculate the required power, the efficiency of self-swimming, and all related forces (both oscillatory and averaged). However the large number of terms in (2.11) makes all these problems rather cumbersome, and places them out of the scope of this short paper.

2. Our approach (based on the two-timing method and distinguished limit) is technically different from all previous studies in this area. The results for  $N$ -sphere swimmer show its analytical strength.

3. The expression (2.11) can be predicted without any calculations, on the base of the result for  $N = 3$ . Indeed, if we are interested in the main term of the order  $\varepsilon^2\delta$ , then only the triple interactions can be taken into account, as they have been described by Golestanian & Ajdari (2008). The additional (to triplets) interactions between four spheres will inevitably produce the next order term  $O(\varepsilon^3\delta)$ , which we do not consider.

4. There are some interesting discussions about the physical mechanism of self-propulsion in the quoted literature. However one can also notice that a similar result does exist for self-propulsion in an inviscid fluid (Saffman (1967)) and some physical explanation can be achieved if we replace the term ‘virtual mass of a dumbbell’ by the term ‘viscous drag coefficient of a dumbbell’. Say, for a three-sphere swimmer this coefficient decreases when the distance between two neighbouring spheres (a dumbbell) decreases and then the third sphere is used to ‘push’ or ‘pull’. If the reverse motion of the third sphere meets the increased drag coefficient of the dumbbell, then self-propulsion is achieved.

5. The mathematical justification of the presented results by the estimation of the error in the original equation can be performed similar to Vladimirov (2010), Vladimirov (2011).

6. One can also derive the higher approximations of self-propulsion velocity, as it has been done by Vladimirov (2010), Vladimirov (2011). They can be especially useful for



the studies of motion with  $\overline{\mathbf{V}}_0 \equiv 0$  (say, if all correlations involved to (2.11) are zero). In this case one can show that self-propulsion can be generated by interactions of four and more spheres.

7. In this paper we consider only periodic oscillations of constraints. The studies of non-periodic oscillations might represent an interesting problem. An attempt in this direction have been made by Golestanian & Ajdari (2009). In fact, such generalizations have been already considered for many different oscillating systems.

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